THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-Office at Kidder, Missouri, as Second-Class Mail Matter.

VOL. I. DECEMBER, 1894. NO. 12.

BIOGRAPHY.

PROFESSOR FELIX KLEIN.

BY DR. GEORGE BRUCE HALSTED.

THE eminent subject of this very imperfect sketch was born on the twenty-fifth of April 1849 in Duesseldorf. His mother was Elise Sophie *née* Kayser; his father, the "Landrentmeister" Caspar Klein, both of the protestant faith. For eight years, from the autumn of 1857 to the autumn of 1865 he attended the Duesseldorf Gymnasium, and went thence to the University of Bonn, for the study of mathematics and the natural sciences, especially physics. Here he had the extraordinary good fortune to come into close relations with the great Professor Pluecker, who gave him the position of assistant in the physical institute of Bonn, and used his help in writing out his profoundly original and stimulating mathematical works.

The death of Pluecker May 22nd 1868 closed this formative period, of which the influence on Klein can not be over estimated. So mighty is the power of contact with the living spirit of research, of taking part in original work with a master, of sharing in creative authorship, that any one who has once come intimately in contact with a producer of the first rank must have had his whole mentality altered for the rest of his life.

The gradual development, high attainment, and then continuous achievement of Felix Klein are more due to Pluecker than to all other influences combined. His very mental attitude in the world of mathematics constantly recalls his great maker.

Of others whose lectures he attended, we may mention Argelander and Lipschitz, to the latter of whom particularly he has expressed his gratitude for kindly and efficient guidance and aid in his studies. Klein took his doctor's

PROFESSOR FELIX KLEIN.

degree at Bonn on December 12th 1868 with a dissertation "On the transformation of the general equation of the second degree between line-coordinates to a canonic form," a subject taken from the analytic line-geometry of his master Pluecker. A line-complex of the *n*th degree contains a triply infinite multitude of straights, which are so distributed in space, that those straights which go through a fixed point make a cone of the *n*th order, or, what is the same, that those straights which lie in a fixed plane envelop a curve of the *n*th class. Such an aggregate or form finds its analytic representation through the coordinates of the straight in space, introduced by Pluecker. According to Pluecker the straight has six homogeneous coordinates which fulfill an equation-of-condition of the second degree. By means of these the straight is determined with reference to a coordinate-tetrahedron. A homogeneous equation of the *n*th degree between these coordinates represents a complex of the *n*th degree.

The dissertation transforms the equation of the second degree between line-coordinates to a canonic form, in correspondence with a change of the coordinate-tetrahedron. It first gives the general formulas to be applied in such a transformation.

From these the problem appears algebraically as the simultaneous linear transformation of the complex to a canonic form, and of the equation-of-condition, which the line coordinates must fulfill, into itself. In carrying out these transformations, it attains to a classification of the complexes of the second degree into distinct species.

The dissertation is dedicated to Pluecker and contains eight specific references to Pluecker's "Neue Geometrie des Raumes, gegruendet auf die Betrachtung der geraden Linie als Raumelement." It is lucid and simple, but for depth and promise contrasts sharply with the great dissertation of Riemann, that "book with seven seals."

It may be interesting, as characteristic of this germinating state, to note that of his five theses the second calls attention to one of Cauchy's slips in logical rigor, slips now known to be so numerous that C. S. Peirce makes of them a paradox, maintaining that fruitfulness of Cauchy's work is essentially connected with its logical inaccuracy.

The third thesis declares the assumption of an ether unavoidable in the explanation of the phenomena of light.

The last thesis is the desirability of the introduction of newer methods in Geometry alongside the Euclidean in gymnasial teaching.

This serves, it seems, to emphasize my point that the long eight years of gymnasial so-called *training* left the seed still dormant, and only in Pleucker did it find the rain and the sun to call it to life and growth.

Within two years now the devolopment is amazing. Already in 1870 he is working with another great genius, Sophus Lie; and in 1871 is presented to the Goettingen Academy of Science his epoch-making paper, "Ueber die sogenannte Nicht-Euklidische Geometrie." Its aim is to present the mathematical results of the non-Euclidean geometry, in so far as they pertain to the

theory of parallels, in a new, intuitive way; its instrument is the mighty projective geometry, which he proves independent of all question of parallels. He perfects the projective metrics of Cayley by founding cross-ratio, after von Staudt, wholly without any use or idea of measurement. Then can be constructed a general projective expression for distance, related to an arbitrary surface of the second degree as Fundamental-surface (Cayley's Absolute). This projective metrics then gives, according to the species of Absolute used, a picture of the results of the parallel-theory in the space of Lobachwsky, of Euclid, of Riemann. But not merely a picture; they coincide to their innermost nature.

The paper begins by stating that, as well-known, the eleventh axiom of Euclid is equivalent to the theorem that the sum of the angles in a triangle equals two right angles. Legendre gave a proof that the angle-sum in a triangle cannot be greater than two right angles; but this proof, like the corresponding one in Lobachevsky, assumes the infinite length of the straight.

Drop this assumption, and the proof falls, else would it apply in surface spherics. Legendre showed further, that if in one triangle the angle-sum is two right angles, it is so in every triangle. We now know that this had been proven long before by Saccheri. But Professor Klein said that he heard the name of Saccheri for the first time in my address before the World's Science Congress. But it is claimed for Gauss that he was the first to distinctly state his conviction of the impossibility of proving the theorem of the equality of the angle-sum to two right angles. But it does not follow, as claimed by his Goettingen worshippers, that Gauss ever came to the conviction that a valid non-Euclidean geometry was possible until after it had been made simultaneously by John Bolyai and Lobachevsky, and perhaps long before by Wolfgang Bolyai. Certainly the world did not hear of it from Gauss. He published nothing on it.

In this non Euclidean geometry there appears a certain constant characteristic for the metrics of the space. By giving this an infinite value we obtain the ordinary Euclidean geometry. But if it has a finite value, we get a quite distinct geometry, in which, for example, the following theorems hold: The angle sum in a triangle is less than two right angles, and indeed so much the more so the greater the surface of the triangle. For a triangle whose vertices are infinitely separated, the angle-sum is zero. Through a point without a straight one can draw two parallels to the straight, that is, lines which cut the straight on the one or the other side in a point at infinity. The straights through the point which run between the two parallels no where cut the given straight. But on the other hand, in Riemann's marvellous inaugural lecture, "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen," is pointed out that the unboundedness of space, which is experiential, does not carry with it the infinity of space.

It is thinkable, and would not contradict our perceptional intuition, which always relates to a finite piece of space, that space is finite and comes back into itself.

The geometry of our space would then be like that of a tridimensional sphere in a four dimensional manifoldness. This representation carries with it that the angle-sum in a triangle, as in ordinary spherical triangles, is greater than two right angles, and indeed the more so, the greater the triangle. The straight would then have no point at infinity, and through a given point no parallel to a given straight could be drawn. Now Cayley constructed his celebrated projective metrics to show how the ordinary Euclidean metrics may be taken as a special part of projective geometry. Klein generalizes Cayley and founds three metric geometries, the elliptic (Riemann's), the hyperbolic (Lobachevsky's), the parabolic, (Euclid's).

This little paper of 1871 contains the promise of much that is most genial in the after work of a man now generally considered as the most interesting and one of the very greatest of living mathematicians. Of all those splended and charming series of lectures with which Klein has made Goettingen so attractive to the whole world, the most delightful and epoch-making are those on non-Euclidean geometry, (Nicht-Euklidische Geometrie, I. Vorlesung, gehalten waehrend des Wintersemesters 1889-90 von F. Klein. Ausgearbeitet von Fr. Schilling. Zweiter Abdruck. Goettingen 1893. Small Quarto, lithographed, pp. v. 365. II. Sommersemesters 1890. Zweiter Abdruck 1893. pp. iv. 238.)

The World's Science Congress at Chicago was in nothing more fortunate than in the presence of Helmholtz and Felix Klein, and in the spontaneous and universal homage accorded them no idea was more often emphasized than their connection with the birth and development of that wonderful new world of pure science typified in the non-Euclidean geometry.

The narrow limits of this feeble sketch prevent the statement of how much promise, richly fulfilled in the development of this many-sided man, in totally other directions is contained in a little-known paper of 1873, "Ueber den allgemeinen Functionsbegriff und dessen Darstellung durch eine willkuerliche Curve."

Twenty years of production and achievement have not in the least dampened the ardour of this enthusiastic mind. This very summer at the great meeting of scientists in Vienna Klein seemed the busyest, the foremost of all that goodly company.

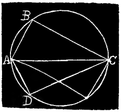
ISOPERIMETRY WITHOUT CURVES OR CALCULUS.

By Professor P. H. PHILBRICK, M. Sc., C. E., Lake Charles, Louisiana.

[Continued from the November Number.]

PROPOSITION VII. If in a quadrilateral two sides are equal, and the

other two sides are also equal; the quadrilateral may be inscribed either with the equal sides adjacent giving $ABCD$, or with the equal sides opposite giving $ABCD'$.

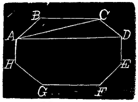


For AC is a diameter of the circumscribing circle in either case.

PROPOSITION VII. To find the polygon of n sides of given perimeter and maximum area.

Let $ABCDEF$ etc. represent the polygon.

Draw the diagonal AC, and suppose the sides AB and BC to vary, while the other sides remain fixed in magnitude and position, and we prove, as in Proposition II, that the area of ABC, and consequently of the polygon is a maximum, when $AB=BC$. Similarly $BC=CD$, $CD=DE$, etc.



Therefore the polygon is equilateral.

Draw the diagonal AD. Then supposing AB, BC and CD to vary in *position*, while the other sides remain entirely fixed, we prove, as in Proposition V, that the angles at B and C are equal. We prove in the same way that the angles are equal at C and D, at D and E, etc; and hence the angles are all equal.

The polygon is therefore both equilateral and equiangular (that is regular) and is inscriptable in a circle.

PROPOSITION VIII. Of all Polygons having the same area and the same number of sides, the regular polygon has the maximum perimeter.

For let P be a regular polygon and M any irregular polygon, having the same number of sides and area.

Now if the perimeter of P was equal to that of M then (Prop. VII) the area of P would be greater than that of M, but since it is equal to that of M its perimeter must be less.

[To be Continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton), Ph. D., (Johns Hopkins), Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the November Number.]

PROPOSITION XII. *Again I say the straight AD somewhere on that side will meet the straight PL (and indeed at a finite, or terminated distance) also in the hypothesis of obtuse angle.*

PROOF. For, as in the preceding proposition, DF being assumed equal to AD; and the just noted perpendiculars let fall, I must show the join DM will be greater than DF, or DA, and therefore (P.X.) the straight BM will be greater than AB.

And in the first place DM will not be equal to DF. Otherwise the angle DMF would be equal (Eu. I. 5.) to the angle DFM, and therefore greater (P.VIII, in the hypothesis of obtuse angle) than the external angle XAH, or its verticle CAF.

Wherefore (since the angles CAM, FMA are taken equal, as being right) the remaining angle DMA would be less than the remaining angle DAM. This is absurd (against Eu.1. 5.), if indeed DM be equal to DF, or DA.

But neither is DM less than DF, or DA. Otherwise (Eu.I.18) the angle DMF would be greater than the angle DFM, and therefore still greater (in the hypothesis of obtuse angle) than the external angle XAH, or its vertical CAD, Wherefore again, as above, the remaining angle DMA would be still less than the remaining angle DAM. But this is absurd (against Eu.I.18) if indeed DM be lesss than DF, or DA.

It remains therefore, that the join DM is greater than DF, or DA, and therefore (P.X.) BM is greater than AB. Quod erat hoc loco intentum.

Since therefore, assuming in AD produced the interval AF double the interval BD, the perpendicular FM let fall on the transversal AP cuts off from it more than double what is cut off by the perpendicular let fall from the point D; more quickly by far in this hypothesis of obtuse angle, than in the preceding hypothesis of right angle, we attain to an interval so great, that from it the perpendicular let fall cuts off a base greater than the designated AP however great.

But this, as in the preceding proposition, could not happen, unless after the meeting of the produced AD with PL in some point; and indeed at a finite, or terminated distance. Quod erat etc.

PROPOSITION XIII. *If the straight XA (of designated length however great) meeting two straights AD, XL makes with them toward the same parts* (fig.11) *internal angles XAD, AXL less than two right angles: I say, these two (even if neither of those angles be a right angle) at length will mutually meet in some point on the side toward those angles, and indeed at a finite, or terminated distance, if either hypothesis holds, of right angle or of obtuse angle.*

PROOF. For one of the said angles, as AXL suppose, will be acute. Accordingly from the vertex of the other angle is dropped the perpendicular AP on XL, which certainly (because of Eu. I. 17.) falls on the side of the acute angle AXL. Since therefore in the triangle APX, right-angled at P, the two acute angles

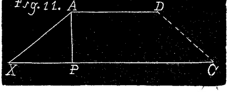


Fig. 11.

PAX, PXA, together are not less (P.IX.) than a right angle, in either hypothesis, of right angle, or of obtuse angle; if these two angles are taken away from the sum of the given angles the then remaining angle PAD will be less than a right angle. Consequently we will be in the case of the two preceding propositions, since it is obvious that one or the other hypothesis holds, either of right angle, or of obtuse angle.

Wherefore the straights AD, and PL, or XL, meet in some point at a finite, or terminated distance on the side noted, as well under the one, as under the other mentioned hypothesis. Quod erat demonstrandum.

[To be Continued.]

THE INSCRIPTION OF REGULAR POLYGONS.

By LEONARD E. DICKSON, M. A., Fellow in Pure Mathematics, University of Chicago.

CHAPTER IV.

[Continued from the November Number.]

We will now consider regular polygons the number of whose sides is a *composite* number. The present chapter will be devoted to the case when n is divisible by 3.

In the regular polygon of 21 *sides*, the chord $A_7=1$; and $A_3-A_6+A_9=1$, being the chords of a regular 7-gon.

But $A_1-A_2+A_3-A_4+A_5-A_6+A_7-A_8+A_9-A_{10}=1$.

Hence, $(A_1-A_8)+(-A_2+A_5)+(-A_4-A_{10})=-1$.

Now $(A_1-A_8)(-A_4-A_{10})=-[(-A_4-A_{10})+(-A_2+A_5)+2(A_3+A_9)]$.

$(A_1-A_8)(-A_2+A_5)=-[(-A_4-A_{10})+(A_1-A_8)+2(A_3-A_6)]$.

$(-A_2+A_5)(-A_2-A_{10})=-[(A_1-A_8)+(-A_2+A_5)+2(A_9-A_6)]$.

The sum of these 3 products $=-2[(A_1-A_8)+(-A_2+A_5)+(-A_4-A_{10})$
$+2(A_3-A_6+A_9)]=-2.$

Again, $(A_1-A_8)(-A_2+A_5)(-A_4-A_{10})=(A_2-A_5)[(-A_4-A_{10})$
$+(-A_2+A_5)+2(A_3+A_9)]=(A_1-A_8)+(-A_2+A_5)+(-A_4-A_{10})$
$+2(A_3-A_6+A_9)=1$; for $(A_2-A_5)(A_3+A_9)=(A_1-A_8)+(-A_2+A_5)$
$+(-A_4-A_{10})+2=1$; and $(A_2-A_5)^2=2+A_4+A_{10}-2A_3$.

$\therefore$ (A_1-A_8), $(-A_2+A_5)$ and $(-A_4-A_{10})$ are the roots of the cubic

$$x^3+x^2-2x-1=0.$$

But it was shown in Chapter I that the chords $A_3, -A_6, A_9$ of the regular 7-gôn are the roots of $x^3-x^2-2x+1=0$.

Thus the two sets of roots are numerically equal but of opposite sign.

$\therefore\ A_1 - A_8 = A_6;\ -A_2 + A_5 = -A_9;\ A_4 + A_{10} = A_3.$

We may write these symmetrically:

$$\begin{cases} A_1 - A_6 - A_8 = 0 \\ A_2 - A_5 - A_9 = 0 \\ A_3 - A_4 - A_{10} = 0. \end{cases}$$

Now $(-A_1.A_6 - A_1.A_8 + A_6.A_8) = (A_2 - A_5 - A_9 - 3A_7) = -3.$

$A_1.A_6.A_8 = A_1(A_2 - A_7) = A_1 + A_3 - A_6 - A_8 = A_3.$

Hence, $A_1, -A_6, -A_8$ are the roots of $x^3 - 3x - A_3 = 0.$

Similarly, $A_2, -A_5, -A_9$ are the roots of $x^3 - 3x - A_6 = 0.$

$A_3, -A_4, -A_{10}$ are the roots of $x^3 - 3x - A_9 = 0.$

In the regular polygon of 27 *sides*, $A_9 = 1$; $A_3 - A_6 + A_9 - A_{12} = 1$. But $A_1 - A_2 + A_3 - A_4 + A_5 - A_6 + \dots + A_{13} = 1.$

$\therefore\ (A_1 - A_8 - A_{10}) + (-A_2 + A_7 + A_{11}) + (-A_4 + A_5 + A_{13}) = 0.$

Now $(A_1 - A_8 - A_{10})(-A_2 + A_7 + A_{11}) = -3(A_1 - A_8 - A_{10}) + 3(A_{12} + A_6 - A_3) = -3(A_1 - A_8 - A_{10})$. Hence, either $(-A_2 + A_7 + A_{11}) = -3$, or $(A_1 - A_8 - A_{10}) = 0$. Again, $(A_1 - A_8 - A_{10})(-A_4 + A_5 + A_{13}) = -3(-A_4 + A_5 + A_{13})$. Hence, either $(A_1 - A_8 - A_{10}) = -3$, or $(-A_4 + A_5 + A_{13}) = 0$. Lastly, $(-A_2 + A_7 + A_{11})(-A_4 + A_5 + A_{13}) = -3(-A_2 + A_7 + A_{11})$. Hence, either $(-A_4 + A_5 + A_{13}) = -3$, or $(-A_2 + A_7 + A_{11}) = 0.$

But $(A_1 - A_8 - A_{10}) + (-A_2 + A_7 + A_{11}) + (-A_4 + A_5 + A_{13}) = 0.$

Hence each term can not equal -3, but must be 0.

$$\therefore \begin{cases} A_1 - A_8 - A_{10} = 0 \\ A_2 - A_7 - A_{11} = 0 \\ A_3 - A_6 - A_{12} = 0 \\ A_4 - A_5 - A_{13} = 0. \end{cases}$$

Now $(-A_1.A_8 - A_1.A_{10} + A_8.A_{10}) = -[3 + (-A_2 + A_7 + A_{11})] = -3.$

$A_1.A_8.A_{10} = A_1(A_2 - A_9) = A_3 + (A_1 - A_8 - A_{10}) = A_3.$

Hence, $A_1, -A_8, -A_{10}$ are the roots of $x^3 - 3x - A_3 = 0.$

Similarly, $A_2, -A_7, -A_{11}$ are the roots of $x^3 - 3x - A_6 = 0.$

$A_4, -A_5, -A_{13}$ are the roots of $x^3 - 3x - A_{12} = 0.$

By induction we derive that for a regular polygon of $n = 3m$ sides:

$$A_1 - A_{m-1} - A_{m+1} = 0$$

$$A_2 - A_{m-2} - A_{m+2} = 0$$

Generally, $A_s - A_{m-s} - A_{m+s} = 0.$

To prove the general formula, throw it into its trigonometric form,

$\cos\frac{s\pi}{3m} - \cos\frac{(m-s)\pi}{3m} - \cos\frac{(m+s)\pi}{3m} = 0$, which follows since

$$\cos\frac{(m-s)\pi}{3m} + \cos\frac{(m+s)\pi}{3m} = 2\cos\frac{\pi}{3}.\cos\frac{s\pi}{3m} = \cos\frac{s\pi}{3m}.$$

Again, $(-A_1 . A_{m-1} - A_1 . A_{m+1} + A_{m-1} . A_{m+1}) = (A_2 - A_{m-2} - A_{m+2}) - 3A_m$
$= -3$, since A_m is the unit radius.

$A_1 . A_{m-1} . A_{m+1} = A_1(A_2 - A_m) = A_1(A_2 - 1) = A_1 + A_3 - A_1 = A_3$.

$\therefore \quad A_1, -A_{m-1}, -A_{m+1}$ are the roots of $x^3 - 3x - A_3 = 0$.

Similarly, $A_2, -A_{m-2}, -A_{m+2}$ are the roots of $x^3 - 3x - A_6 = 0$.

Generally, $A_s, -A_{m-s}, -A_{m+s}$ are the roots of $x^3 - 3x - A_{3s} = 0$,

where s is and integer $\leqq \frac{m-1}{2}$; and $A_3, -A_6, A_9, \ldots (-1)^{s-1} A_{3s}, \ldots\ldots$ are the $\frac{m-1}{2}$ roots of the general equation (4) of Chapter II. for the regular m-gon.

If m is prime to 3, i.e., if n contains the factor 3 in but the first degree, one chord out of every group of three chords given above is a root of this equation. For, m being prime to 3, one and only one of the subscripts s, $m-s$, $m+s$ is disisible by 3, as is seen by writing them respectively $3m-s$, $m-s$, $2m-s$.

The remaining two chords in the group will be roots of a quadratic whose coefficients are linear functions of the roots of equation (4).

Thus, if $m+s$ be divisible by 3, $m-2s$ will also.

Then $A_s - A_{m-s} = A_{m+s}$. $A_s . A_{m-s} = A_m + A_{m-2s} = 1 + A_{m-2s}$.

Hence, A_s and $-A_{m-s}$ are the roots of $x^2 - A_{m+s}x - (1 + A_{m-2s}) = 0$, in which A_{m+s} and A_{m-2s} are roots of the equation (4) for the regular m-gon. Hence, if the $\frac{m-1}{2}$ chords of the regular m-gon be found, we can find all the chords of the regular $3m$-gon by solving a series of quadratics.

However, if m is divisible by 3, i.e., if n contains the factor 3^2, the three chords in each of the above groups must either all or none be roots of (4); for the subscripts s, $m-s$, $m+s$, are then either all or none divisible by 3 according as s is divisible by 3 or not. Hence we can neither lower the cubics nor avoid them.

The regular 3m-gon depends, therefore, for inscription upon the same equations as does the regular m-gon, if m be prime to 3; but depends upon one or more cubics in addition to the former equations, if m contains the factor 3.

ERRATA in Chapter III p 376 line 1, *for* form *read* degree p 377 lines 1 and 5, extend radical sign over 2 $(17 \pm \sqrt{17})$, line 14, *for* $(-A_1 - 2A_2 + \ldots)$ *read* $-(A_1 - 2A_2 + \ldots)$, line 18, seperate the two products thus] ; (, line 21, *for* $2b$ read $2B$.

[To be Continued.]

PASCAL'S ARITHMETICAL TRIANGLE.

By GEORGE LILLEY, Ph. D., LL.D., Ex-President of Washington State Agricultural College and School of Science, Portland, Oregon.

I do not remember of having ever seen an account of this interesting device in any of our American text books, and, so far as I am able to ascertain, it has not been published in this country. The accompaning diagram explains itself.

1	1	1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10	11	
1	3	6	10	15	21	28	36	45	55		
1	4	10	20	35	56	84	120	165			
1	5	15	35	70	126	210	330				
1	6	21	56	126	252	462					
1	7	28	84	210	462						
1	8	36	120	330							
1	9	45	165								
1	10	55									
1	11										
1											

To find any number, in a triangle, take the sum of the number immediately above and the number immediately to the left of the required number, or take the sum of the numbers immediately above and to the left of the required number. Thus, the 7th number in the 4th row $=28+56=84$, or

$$28+21+15+10+6+3+1=84.$$

The numbers on the diagonals are the coefficients of the expansion of a binomial.

The mth number in the nth row is given by the formula

$$\frac{|m+n-2}{|(m-1)|(n-1)}.$$

Thus, the 7th number in the 5th row

$$=\frac{|7+\;-2}{|(7-1)|(5-1)}=\frac{|10}{|6\;|4}=\frac{1.2.3.4.5.6.7.8.9.10}{1.2.3.4.5.6.1.2.3.4}=210., \text{ etc.}$$

LOGICAL DEDUCTIONS FROM THE HYPOTHESIS THAT THE ANGLE-SUM IS LESS THAN TWO RIGHT ANGLES.

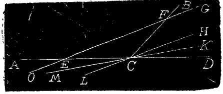
By Professor JOHN N. LYLE, Ph. D., Professor of Natural Science, Westminster College, Fulton, Missouri.

Let ECF be any individual rectilineal triangle whatever whose angle sum is assumed to be less than two right angles.

Produce the side EC to D. Then $ECF+FCD=$Two right angles. (Prop. XIII. Book I. of Euclid's Elements.) But by hypothesis $FEC+EFC+ECF<$Two right angles. Therefore $ECF+FCD>FEC+EFC+ECF$. Take ECF from both members.

Then in accordance with Euclid's axiom 5, $FCD>FEC+EFC$.

That is, one logical deduction from our hypothesis is that the exterior angle FCD is greater than the sum of the two interior and opposite angles FEC and EFC.



Lay off $FCH=EFC$.

Since $FCD>FCH$, CH must lie between CF and CD. Since FCH is equal to EFC, the line CH is parallel to the line EF according to proposition XXVII. Book I. of Euclid's Elements.

Euclid gives the following definition:

"Parallel straight lines are such as, being in the same plane, do not meet however far they are produced in both directions."

Since the lines CH and EF are parallel in the Euclidian sense they do *not* meet. Is the assumption *that there is a point* at which the two parallel lines meet "an extension" or a contradiction of Euclid's doctrine? Is it as difficult to accept as true Euclid's axiom 12 as the statement that there is a point at which two parallel straight lines meet? What Euclid calls his axiom 12 is regarded as a sound proposition although it may not be self-evident. *The statement that two parallel straight lines meet at a point called "infinity" is certainly not axiomatic and just as certainly has never been demonstrated.*

Again, lay off the angle $KCD=FEC$.

Since $KCD=FEC$, the line CK is parallel to the line EF according to Proposition XXVIII. Book I. of Euclid's Elements.

Since the lines CK and EF are parallel in the Euclidian sense they do *not* meet.

Since by inference from hypothesis $FCD>FEC+EFC$ and by construction $FCH+KCD=EFC+FEC$ it follows that $FCD>FCH+KCD$. But if FCD is greater than $FCH+KCD$, HCK must be a *real* angle.

It must be not only a *real* angle, but *an individual* angle; for according to the hypothesis from which we set out, the triangle ECF is an individual one, and hence HCK must be an individual angle.

It follows also that HCK, since it has but one value only for the same individual triangle ECF, can not be a variable. Further, if HCK is a real angle, it follows that HC and KC are seperate and distinct lines, although shown by logical deduction from the initial hypothesis to be parallel to the same line EF in the sense that they do not meet it when produced. The conclusion that the straight lines HC and KC meeting at C are both of them parallel to the straight line EF contradicts the statements known as Pleyfair's axiom that "Two straight lines, which intersect one another, can not be both parallel to the same straight line."

If the angle sum is two right angles, it readily follows that the ex-

terior angle FCD is equal to the sum of FCH and KCD and that the lines CH and CK are coincident. This deduction is contradicted by the conclusion that CH and CK are separate and distinct lines deduced from the hypothesis that the angle sum is less than two right angles.

Are the lines CH and CK the boundry lines between the *cutting* and the *not cutting* lines referred to in Lobatschewsky's theory of parallels, theorem 16?

CH as we have already seen is parallel to EF in the Euclidian sense; that is, these two lines will not meet however far they may be produced both ways.

CH can not be one of Lobatschewsky's boundary lines through C for they are supposed by him *gradually to approach and ultimately to meet* EF.

It was deduced from the hypothesis that CK, also, is parallel to EF in the Euclidian sense, and hence can not be one of Lobatschewsky's boundary lines through C.

If CH and CK are Lobatschewsky's boundary lines an inconsistency in his "Imaginary Geometry" emerges for his boundary lines through C ultimately meet EF whereas CH and CK can not do so without discrediting Propositions XXVII and XXVIII of Euclid's Elements. From this point of view Lobatschewsky seems to contradict inferences logically obtained from the hypothesis that the angle sum is less than two right angles.

Lobatschewsky's System of geometry is made up of two parts, one of which is distinctively Euclidian and the other, distinctively anti-Euclidian called by him "Imaginary Geometry". The "Imaginary Geometry" part flatly contradicts the Euclidian part. If CH and CK are Lobatcshewsky's boundary lines his "Imaginary Geometry" is seen to be inconsistent with itself.

The hypothesis of Lobatschewsky that the angle sum of a rectilineal triangle is less than two right angles contradicts the hypothesis of Euclid that it is equal to two right angles. According to the logical law of excluded middle, if one of two propositions that mutually contradict each other is true, the other must be false.

Let a straight line drawn in the plane of the triangle ECF through the point C be revolved about that point, will it occupy more than one position in which it does not cut the line EF?

The answer *no* is returned by those geometers who accept two right angles as the angle sum of a rectilineal triangle. With them CH and CK are one and the same straight line.

The answer *yes* must be the logical reply of those who accept less than two right angles as the angle sum of a rectilineal triangle and who also accept the first twenty-eight propositions of Book I., Euclid's Elements. With them CH and CK must be regarded as separate and distinct lines although neither is supposed to cut EF in any point whatever.

The answer *yes* is given by Lobatschewsky. His doctrine is that there is through C a pencil of not-cutting lines between two boundary lines that

untimately cut the line EF. These hypothetical boundary lines are called by him paralles.

These Lobatschewsky parallels are not Euclidian parallels, since according to Euclid's geometry but one straight line can be drawn through C in the plane of EF that does not cut EF.

Are the Lobatschewsky parallels in the same plane of those of Euclid or are they supposed to be on a pseudo-spherical surface? Lobatschewsky expressly says "rectilineal triangle." But a rectilineal triangle can not be drawn on a curved surface. Some of the followers of Lobatschewsky informs us, however, that the surface of Lobatschewsky's is not a Euclidian plane.

Is our Universe located in *one* space only or has Lobatschewsky discovered another and a different space to contain it -- a space from which Euclid's 12th axiom is banished, where indeed the angle sum of a triangle is forever less than two right angles, where the strange device—"*Hypothesis Anguli Acuti*"—is emblazaned on all banners, where there are no planes but pseudo-spherical surfaces instead having negative curvature, and where there are no straight lines but lines lost eternally to rectitude.

For untold ages it has been believed that there is but *one* space, that in which the material bodies of the Universe are contained and through which they move in describing their orbits. This belief of the ages is contradicted by the Kantian Idealists who maintain that space is purely subjective, that is, that it is the product of and exists only in the human mind.

The mind of man cognizes but does not create space.

This belief of the ages referred to above is, also, discredited by Kantian agnostics and nihilists who doubt or deny the truth of the report made by our intelligence that material bodies occupy space and are contained in space There has from the outset been an antagonism between the science of Physics and Kantian idealism, Nihilism and agnosticism.

The space in which our universe is contained is currently believed to be trinally extended. By that is meant that three straight lines only mutually perpendicular to each other can be drawn through any point.

If space is *extended* it must be objective to the human mind.

In *objective space* straight and curved lines, planes and curved surfaces, and volumes such as cubes, cones and spheres may be located.

To call a line a space of one dimension; and a surface, a space of two dimensions is an absured use of language. Curved surfaces are found in space. To charteterize *such surfaces* as *new species of space* is to subject the meaning of the word space to a strange, grotesque and irrational Metamorphosis.

We close this article with the remark that Euclid in his *Reductio ad absurdum* process argues from false premises in order to show that they are false and Lobatschewsky in his Theory of Parallels argues from false premises with such plausibility and subtle sophistry as to allure many into accepting both premises and conclusions as sound geometry.

ARITHMETIC.

Conducted by B. F. FINKEL. Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

31. Proposed by I. L. BEVERAGE, Monterey, Virginia.

"A man wishes to know how many hogs at \$9, sheep at \$2, lambs at \$1, and calves at \$9 per head, can be bought for \$400, having of the four kinds, 100 animals in all. How many different answers can be given?"

[Satisfactory arithmetical solution desired.]

Solution by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Kidder Institute, Kidder, Missouri.

The average price per animal $=\$400\div100=\$4.00$.

		Dif.	Balance	Columns									
	\$1	\$3	5 lambs	5 sheep	3	10	17	24	31	38	45	52	59
\$4	\$2	\$2	3 hogs & calves	2 hogs & calves	68	60	52	44	36	28	20	12	4
	\$9	\$5			29	30	31	32	33	34	35	36	37
			8	7									

For convenience of expression suppose the animals were bought at the prices named and sold at the average price \$4. Then a lamb bought for \$1 and sold for \$4 is a gain of \$3; a sheep bought for \$2 and sold \$4 is a gain of \$2; a hog bought for \$9 and sold for \$4 is a loss of \$5; and a calf bought for \$9 and sold for \$4 is a loss of \$5. The gains on the sheep and lambs must be balanced by the losses on the hogs and calves. The L. C. M. of \$3 and \$5 is \$15. If we gain \$3 on one lamb to gain \$15 we must take as many lambs as \$3 is contained in \$15, which are 5 lambs. If we lose \$5 on a hog and a calf, to lose \$15 we must take as many hogs and calves as \$5 is contained in \$15, which are 3 hogs and calves. In like manner, we find that we must take 5 sheep and 1 hog and 1 calf. Adding the balance columns, considering them as abstract numbers, we have 8 and 7. $8+7=15$. $100\div15=6\frac{2}{3}$. Multiplying the balance columns by $6\frac{2}{3}$ and adding the results horizontally, we get $33\frac{1}{3}$ lambs, $33\frac{1}{3}$ sheep and $33\frac{1}{3}$ hogs and calves, a result incompatible with the nature of the problem. Now it is clear that we must take a certain number of 8's + a certain number of 7's to make 100. Arithmetically, this can be done by trial only. We find by trial that two 8's and twelve 7's make 100. Hence, multiplying the column of 8 by 2 and the column of 7 by 12 and add the results horizontally we get 10 lambs, 60 sheep, and 30 hogs and calves. In like manner, we find that nine 8's and four 7's make 100. Multiplying the column of 8 by nine and the column of 7 by four, and adding the results horizontally we get 45 lambs, 20 sheep, and 35 hogs and calves. These are the only results that can be obtained by multiplying the balance columns by integral numbers. Since the numbers in the balance columns, when added horizontally give 5 as results, we may find a number of *fifths* times 8 and a number of fifths times 7

to make 100. By trial we find that $\frac{3}{5}\times 8+\frac{6\frac{8}{5}}{}\times 7=100$. Hence, multiplying the column of 8 by $\frac{3}{5}$ and the column of 7 by $\frac{68}{5}$ and adding the results horizontally, we get 3 lambs, 68 sheep, and 29 hogs and calves. In like manner, by multiplying the column of 8 by $\frac{10}{5}$ and the column of 7 by $\frac{60}{5}$, we get 10 lambs, 60 sheep, and 30 hogs and calves. By multiplying by $\frac{17}{5}$ and $\frac{52}{5}$ respectively, we get 17 lambs, 52 sheep, and 31 hogs and calves. The remaining six answers may now be easily written down without further trial. Since, in the first result there are 29 hogs and calves, we may have 1 hog and 28 calves, and so on. In all we may have 28 different results. In like manner in the second result, we may have 29 different results. In like manner 30 in third, 31 in fourth, 32 in the fifth, 33 in the sixth, 34 in the seventh, 35 in the eighth and 36 in the ninth. Hence, in all, we have $28+29+30+31+32+33+34+35+36=288$ different results.

For a fuller treatment of this class of problems see my *Mathematical Solution Book*.

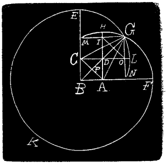
Correct solutions to this problem were received from *H. J. Geartner*, *Martin Spinx*, *John McDowell*, *P. S. Berg*, *P. C. Cullen*, *and M. A. Gruber*. *Mr. Gruber also gave a solution by Algebra.*

32. Proposed by P. C. CULLEN, Meade, Nebraska.

A horse is tied to a corner of a building 40 feet square, by a rope 110 feet long. Over how much land can he graze?

Solution by MARTIN SPINX, Wilmington, Ohio; Professor A. J. LILLY, Algona, Iowa; and Professor H. J. GEARTNER, Wilmington, Ohio.

Let $ABCD$ represent the barn, side $AB=40$ feet; BE represent the rope, length 110 feet; and $CG=70$ feet. Then the area of $FBEKF=\frac{3}{4}\pi\times 110^2=28510.02$ sq. feet. The area of the two equal quadrants $AFGH$ and $CLGE=\frac{1}{2}\pi 70^2=7696.84$ sq. feet. Now $AC=40\sqrt{2}$ feet; $PG=\sqrt{(GC^2-PC^2)}=\sqrt{(70^2-800)}=10\sqrt{41}$ feet; and $DG=10\sqrt{41}-20\sqrt{2}=10[\sqrt{41}-2\sqrt{2}]$ feet. $\therefore$ Area of square $ODIG=GD^2\div 2=\{10[\sqrt{41}-2\sqrt{2}]\}^2\div 2=638.922932$ sq. ft. The height of the segment GNL is $LO=LD-OD=30$ feet$-\frac{1}{2}\sqrt{2}\,GD=[30-(5\sqrt{82}-20)]$ feet$=5[10-\sqrt{82}]$ feet and the base is $GN=2GO=\sqrt{2}GD=10[\sqrt{82}-4]$ feet. Hence, according to the rule, p. 389, Ray's Higher Arithmetic, the area of the segments GNL and GMH is $2(LO^3\div 2GN+\frac{2}{3}LO\times GN)=2\{[5(10-\sqrt{82})]^3\div 20[\sqrt{82}-4]+\frac{2}{3}\times 5(10-\sqrt{82})\times[10\sqrt{82}-4]\}=320.443888$ sq. ft. and the area of the half segments$=160.221944$ sq. ft. Hence, the area over which the horse grazes$=\frac{3}{4}FBEKF+FAH+LCE-(LOG+ODIG+GIH)=\frac{3}{4}\pi 110^2+\frac{1}{2}\pi 70^2-(160.221944+638.922932)=35407.72514$ sq. ft.



This problem was solved in different ways and with different results by *P. S. Berg*, *John McDowell*, *G. B. M. Zerr*, *and D. G. Durrance*.

[For a demonstration of the rule referred to above, see my *Mathematical Solution Book*, pp. 201 and 202.—Ed.]

PROBLEMS.

40. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the market-price of $m=3\frac{1}{3}\%$-stock, in order that it may yield $n=3\frac{1}{3}\%$ interest after deducting $d=\$\frac{7}{20}$ from every $S=\$12$.

41. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

If I gain \$2 in \$5 by selling a horse for \$150, what per cent. would I gain by selling the horse for \$120?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

26. Comment by WOOSTER W. BEMAN, A. M., Professor of Mathematics, University of Michigan, Ann Arbor, Michigan.

Refering to solution 1, of problem 26, p. 351 of MONTHLY:—In general, a, b, and c must be positive. The inequality $a^2b^2c^2>(a+b-c)^2(b+c-a)^2(c+a-b)^2$ is not a legitimate consequence of the three preceding inequalities, $a^2>a^2-(b-c)^2$, $b^2>b^2-(c-a)^2$, and $c^2>c^2-(a-b)^2$.

30. Proposed by C. A. ROBERTS, Long Bottom, Ohio.

Find the sum of $n=10$ terms of the series $1+15+55+134+265\ldots.$

Solution by H. C. WHITAKER, B. S., C. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania; A. H. BELL, Hillsboro, Illinois; and P. S. BERG, Apple Creek, Ohio.

The first terms in the given series and in the successive orders of differences are 1, 14, 26, 13, 0; whence the sum

$$=n+14\frac{n(n-1)}{1.\ 2}+26\frac{n(n-1)(n-2)}{1.\ 2.\ 3}+13\frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}$$

$$=10+630+3120+2730=6490.$$

Also solved by *B. F. Burleson, D. G. Dorrance, J. F. W. Scheffer, G. B. M. Zerr, and the Proposer.*

31. Proposed by D. G. DORRANCE, Camden, New York.

Sum the series 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, etc. to n terms. Also what is the nth term?

1. Solution by J. W. NICHOLSON, A. M., LL. D., President and Professor of Mathematics, Louisiana, State University and Agricultural and Mechanical College, Baton Rouge, Louisiana.

The series is evidently the sum of the following series:

$$\begin{array}{lcccccccccc}
(a) & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \;\ldots \\
(b) & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \;\ldots \\
(c) & & & & & 1 & 3 & 6 & 10 & 15 & 21 \;\ldots \\
 & & & & & & & 1 & 4 & 10 & 20 \;\ldots \\
 & & & & & & & & & 1 & 5 \;\ldots \\
 & & & & & & & & & & \;\ldots
\end{array}$$

The nth term of (a) is 1; the $(n-2)$th term of (b) is $n-2$; the $(n-4)$th term of (c) is $\frac{(n-3)(n-4)}{2}$; the $(n-6)$th term of (d) is $\frac{(n-4)(n-5)(n-6)}{6}$; etc.

Therefore the nth term of the given series is

$$1+(n-2)+\frac{(n-4)(n-3)}{2}+\frac{(n-6)(n-5)(n-4)}{2.3}+\frac{(n-8)(n-7)(n-6)(n-5)}{2.3.4}$$

$+\ldots$ to 0.

Again, the sum of n terms of (a) is n, of $n-2$ terms of (b) is

$$\frac{(n-2)(n-1)}{2}\text{, of } n-4 \text{ terms of } (c) \text{ is } \frac{(n-4)(n-3)(n-2)}{2.\ 3}\text{, etc.}$$

Therefore, the sum of n terms of the given series is

$$n+\frac{(n-2)(n-1)}{2}+\frac{(n-4)(n-3)(n-2)}{2.3}+\frac{(n-6)(n-5)(n-4)(n-3)}{2.3.4}+\ldots\text{to } 0.$$

PROBLEMS.

42. Proposed by ALEXANDER MACFARLANE, A. M., Sc. D., LL. D., Cornell University, Ithaca, New York.

There are p electors and q candidates for r seats. Each elector has r votes, and he may distribute them as he pleases among the candidates. Find in how many different ways the voting may result, that is, the number of possible states of the poll.

43. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

Four men, A, B, C, and D, start from the same place, the traveling rates of A and C are equal. and the traveling rates of B and D were as 17 to 18, respectively: B could travel one mile in 7 minutes and 12 seconds. A traveled *due* west a certain distance, B traveled *due* north the cube of A's distance *plus* his distance; C traveled *due* east a certain distance, and D traveled *due* south the cube of C's distance *plus* his distance: They all change directions, and A traveled *due* north a certain distance. B traveled *due* east the 5th power of A's distance north; C traveled *due* south a certain distance, and D traveled *due* west the 5th power of C's distance south, when it was found that the sum of the north and south distances traveled by B and D was 351090 feet, and the sum of the distances B and D traveled east and west was 5929200000 feet, and that the product of the distances that A and C traveled east and west *plus* the square of the difference of these distances, *plus one* was 3901; and

that the product of the distances that A and C traveled north and south *plus* the square of the difference *squared*, plus the product multiplied by the square of the difference, was 49410000 [equal to the following new formulas: $(nn+d^2+1)=3901$, and $\{(nn+d^2)^2+(nn\times d^2)\}=49410000$]. How far on a line is each party from the starting place, and how long did it require for B and D each to make the entire trip from starting place to the end?

[The Proposer says:"A city lot at St. Andrews,Florida,will be given to the party sending the EDITOR the first correct answer to the above problem-No.43].

GEOMETRY.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

25. Proposed by L. B. FRAKER, Weston, Ohio.

The sides of a quadrilateral board are $AB=7$ inches, $BC=15$ inches, $CD=21$ inches, and $DA=13$ inches; radius of inscribed circle is 6 inches. (1) What are dimensions of the largest rectangular board that can be cut out of the given board, (2) largest square, (3) largest equilateral triangle? (Please solve without use of the calculus.)

II. Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

From the solution of this problem on page 354, No. 10, Vol., I., we find for another set of values for the angles, $A=112°\ 37'\ 11''$, $B=126°\ 52'\ 12''$, $C=53°\ 7'\ 38''$, $D=67°\ 22'\ 49''$. This demonstrates that AB and CD are parallel. Since CD is the longest side and also parallel to AB, one side of the rectangle and square will coincide with DC. Let KI and HI be the sides of the rectangle.

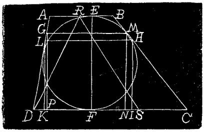
Then $\frac{1}{2}(AB+DC)\times EF=KI\times HI+\frac{1}{2}(AB+KI)(EF-HI)+\frac{1}{2}HI(DK+IC)$.

$\therefore\ \frac{1}{2}(7+21)\times 12=KI\times HI+\frac{1}{2}(7+KI)(12-HI)+\frac{1}{2}(HI)(21-KI)$.

$\therefore\ 126=6KI+7HI$. For a maximum $6KI=7HI$.

$\therefore\ HI=9$ inches, $KI=10\frac{1}{2}$ inches.

$\therefore\ GHIK$ is the rectangle.



For the square $KI=HI$.

$\therefore\ 126=13KI$, $\therefore\ KI=9.692+$inches.

$\therefore\ LMNP$ is the square.

For the triangle, draw DR making the angle $RDC=60°$.

Then $DR^2=EF^2+\frac{1}{4}DR^2$.

$\therefore\ \frac{3}{4}DR^2=144$, $DR^2=192$, $DR=13.856$ inches.

$\therefore\ DRS$ is the triangle.

The above solution is the result of suggestions from the Proposer.

31. Proposed by Professor G. I. HOPKINS, Manchester, New Hampshire.

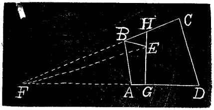
A field is bounded as follows: N. 14° W. 15.2 chains; N. 70° 30′ E. 20.43 chains; S. 6° E. 22.79 chains; N. 86° 30′ W. 18 chains. A spring within it bears from the second corner S. 75° E. 7.9 chains. It is required to cut off 10 acres from the west side of the field by a straight fence through the spring. How far will it be from the first corner to the point at which the division fence meets the fourth side?

Solution by Wm. B. TIMMANS, Professor of Mathematics, St. Mary's College, St. Mary's, Ky.

Let CB and DA be extended to F, and E joined with F.

The bearings give the angles as follows:

$ABF=84°\ 30'$; $BAF=72°\ 30'$; $AFB=23°$; $EBF=145°\ 30'$.



I find $AF=38.722$; $BF=37.101$; and twice area of $ABF=561.34$ sq. ch. In $\triangle EBF$ we have two sides and the included angle; from which I find $EF=43.8411$ ch.; the angle $BFE=5°\ 51'\ 29''$, and the angle $EFG=17°\ 08'\ 31''$.

Let GH be the required line; and call $FH=x$, and $FG=y$.

Hence, we have $xy\sin 23°=561.34+200.=761.34=$twice area GFH....(1).
$43.8411\,x\sin 5°\ 51'\ 29''+43.8411\,y\sin 17°\ 08'\ 31''=761.34=$twice area GFH....(2).

From (1), $xy=\dfrac{761.34}{\sin 23°}=1949.398$, and $x=\dfrac{1949.398}{y}$....(3).

Substituting in (2) and reducing, I get $y^2-58.946\,y=-675.0486$,
$y^2-58.946\,y+(29.473)^2=-675.0486+868.672=193.6234$,

$$y-29.473=\pm 13.9148.$$

Using the *plus* sign (the *minus* sign cannot be used, as it would make G fall to the *left* of A), we have $y=FG=43.3878$; and $AG=4.6658$. *Ans.*

I find $BH=7.828$; $GH=17.672$; $FGH=83°\ 24'\ 09''$; $FHG=73°\ 35'\ 51''$.

Hence the *ten acre lot* is bounded as follows:

N 14° W.	15.20 ch.
N 70½ E.	7.828
S 3° 05′ 51″ E.	17.672
N 86½ W.	4.6658.

Solutions to this problem were received from *G. B. M. Zerr, William E. Kern, H. C. Whitaker, P. H. Philbrick, J. M. Colaw, and A. H. Bell.*

PROBLEMS.

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35. Proposed by LEONARD E. DICKSON, M. A., Fellow in Mathematics, The University of Chicago.

Determine the equation of lowest degree (cubic) upon which depends the inscription of the regular polygon of 37 sides.

36. **Proposed by O. W. ANTHONY, Mexico, Missouri.**

From two points, one on each of the opposite sides of a parallelogram, lines are drawn to the opposite vertices. Through the points of intersection of these lines a line is drawn. Prove that it divides the parallelogram into two equal parts.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

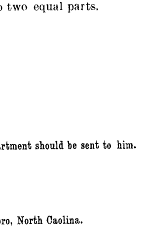
18. **Proposed by J. M. BANDY, A. M., Principal of High School, Ashboro, North Caolina.**

If the ordinate ST of any point T on a circle

$$x^2+y^2=r^2$$

be produced so that $ST.TP=r^2$, prove that the whole area between the locus of P and its asymptotes is double the area of the circle.

III. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Mississippi.



If the ordinate is produced in both directions (upward and downward) there will be two points, P and P_1, located for every point, T, on the circle; and when T has moved over the upper half of the circle, P has traced the upper, and P_1, the lower curve.

x and y being the co-ordinates of P, $ST=\sqrt{(r^2-x^2)}$.

For the upper curve, $TP=y-\sqrt{(r^2-x^2)}$, and the equation is $y_1=\dfrac{2r^2-x^2}{\sqrt{(r^2-x^2)}}$, y_1 being always positive.

For the lower curve, $TP_1=y_1+\sqrt{(r^2-x^2)}$, and the equation is $y_1=\dfrac{x^2}{\sqrt{(r^2-x^2)}}$, y_1 being always negativ.

Therefore the length of $PP_1(=PS+SP_1=y+y_1)$ is $\dfrac{2r^2}{\sqrt{(r^2-x^2)}}$.

Total area between the curves and two common asymptotes $(x=\pm r)$ $=2\displaystyle\int_0^r\frac{2r^2dx}{\sqrt{(r^2-x^2)}}=\left[4r^2\sin^{-1}\frac{x}{r}\right]_0^r=2\pi r^2$.

[The solutions previously published supposed an alteration to be made in the original statement of the problem. Prof. Hume thinks possibly the Proposer had the above in mind. We publish for comparison. —EDITOR.]

24. Proposed by C. W. M. BLACK, A. M., Department of Mathematics, Wesleyan Academy, Wilbraham, Massachusetts.

At the President's reception, the people are admitted at $12\frac{1}{2}$ P. M.: but the *line* in front of the gate, begins to form at 11 A. M. By the time the doors are opened, there are in line 5400 citizens, who have gathered at a rate per second proportional to the time after 11 A. M. The President receives the citizens at the uniform rate of 45 per minute. At what time after 11 A. M. should a citizen *join the line*, in order that he may be *delayed the least* by the reception?

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland, and the PROPOSER.

Let $T,=5400$, seconds $=$ the *line-formation* time; $P,=5400$, persons $=$the number of citizens in line at the expiration of the *first* second; $x=$the number of seconds after 11 A. M. a citizen should join the line, in order to suffer the *least delay*; and $U=$the number of seconds a citizen joining the line, would be delayed by the reception. The number of citizens joining the line, per second, forms during the time T, an arithmetical series in which $a=P_1$, $n=T$, $d=P_1$, $l=a+(n-1)d=TP_1$, and $s=\frac{1}{2}n(a+l)=\frac{1}{2}T(1+T)P_1=P$. $\therefore$ $P_1=2P/T(T+1)$; and consequently, the number of citizens who joined the line during the xth second, is $P_1x=2Px/T(T+1)$. Hence the number of citizens in line at the expiration of the xth second, is

$$N=\frac{x(P_1+P_1x)}{2},=\left(\frac{x(x+1)}{T(T+1)}\right)P\ldots.(1).$$

Let $R=$the number of citizens the President receives per *minute*; then it will take 60 (N/R) *seconds* to receive N citizens. Since the citizens who will actually suffer the least delay has already been delayed $(T-x)$ seconds (waiting in line), the expression for the *total delay*, in seconds, becomes

$$U=(T-x)+\frac{60}{R}\left(\frac{x(x+1)}{T(T+1)}\right)P=\text{a maximum}\ldots\ (2).$$

$$\therefore\ \frac{dU}{dx}=-1+\frac{60}{R}\left(\frac{2x+1}{T(T+1)}\right)P=0\ldots.(3).$$

and $x=\frac{1}{2}\left[\frac{R}{60}\left(\frac{T(T+1)}{P}\right)-1\right],=2024\frac{5}{8}$ seconds; that is, the proper time for joining the line is 33 minutes $44\frac{5}{8}$ seconds after 11 A.M. The total delay occasioned by the reception, is $U=1$ hour 13 minutes $7\frac{13}{16}$ seconds.

Also solved by *C. E. White* and *G. B. M. Zerr*.

PROBLEMS.

32. Proposed by J. F. W. SCHEFFER, Hagerstown, Maryland.

Suppose it to be possible to perform the passage through the north pole: at what latitude would the maximum distance be saved by a ship sailing on the arc of a great circle instead of a parallel of latitude, the points of departure and destination being 180° apart? Also find the maximum saving.

33. Proposed by WILLIAM SYMMOND, A M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, California.

Show that of all curves of a given length, traced on one plane between two given points, and made to revolve around a common axis situated in that plane, the Catenary generates a minimum area.

MECHANICS.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

12. Proposed by J. F W. SCHEFFER, A. M. Hagerstown, Maryland.

A horizontal table without weight is supported on three points, A, B, and C. A weight W is laid upon the table, at a point G. If $AG=a$, $BG=b$, $CG=c$, $\angle AGB=\alpha$, $\angle BGC=\beta$, and $\angle CGA=\gamma$, find the pressures upon A, B, and C.

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

The momental equations with respect to AG, BG, and CG, are respectively:

$$\mathsf{P}_B\times b\sin(\pi-\alpha)=\mathsf{P}_C\times c\sin\beta.\ \therefore\ \frac{\mathsf{P}_B}{\mathsf{P}_C}=\frac{c\sin\beta}{b\sin\alpha}\ldots.(m).$$

$$\mathsf{P}_C\times c\sin(\pi-\beta)=\mathsf{P}_A\times a\sin[\gamma-(\pi-\beta)].\quad\therefore\ \mathsf{P}_C=\left(\frac{a\sin\alpha}{c\sin\beta}\right)\mathsf{P}_A\ldots.(n).$$

$$\mathsf{P}_A\times a\sin(\pi-\gamma)=\mathsf{P}_B\times b\sin[\alpha-(\pi-\gamma)].\quad\therefore\ \mathsf{P}_B=\left(\frac{a\sin\gamma}{b\sin\beta}\right)\mathsf{P}_A\ldots.(p).$$

Put $K=\left(\frac{\sin\beta}{a}+\frac{\sin\gamma}{b}+\frac{\sin\alpha}{c}\right)$; then from the equation, $\mathsf{P}_A+\mathsf{P}_B+\mathsf{P}_C=W$, we deduce the following *symmetrical* and elegant results:

$$\mathsf{P}_A=\left(\frac{\sin\beta}{aK}\right)W,\ \mathsf{P}_B=\left(\frac{\sin\gamma}{bK}\right)W, \text{ and } \mathsf{P}_C=\left(\frac{\sin\alpha}{cK}\right)W.$$

Second Solution.

According to the logic of common-sense, why not write

$$\mathsf{P}_A=\left(\frac{\triangle BCG}{\triangle ABC}\right)W,\ \mathsf{P}_B=\left(\frac{\triangle CAG}{\triangle ABC}\right)W, \text{ and } \mathsf{P}_C=\left(\frac{\triangle ABG}{\triangle ABC}\right)W;$$

and then with the heavy artillery of the higher mathematics *successfully defend* our position ?

NOTE:—These two solutions are to take the place of the *first* solution of this problem published in the November MONTHLY.—*F. P. M.*

PROBLEMS.

18. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Mississippi.

An elliptic paraboloid whose equation is $\frac{y^2}{a}+\frac{x^2}{b}=2z$ has its axis vertical and vertex downward. If μ be the co-efficient of friction, prove that a heavy particle will rest at any point of the surface below its intersection with the cylinder $\frac{y^2}{a^2}+\frac{x^2}{b^2}=\mu^2$.

19. Proposed by H. C. WHITAKER, B. Sc., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

"There was an old woman tossed up in a basket,
Ninety times as high as the moon."

What was her initial velocity, the resistance of the air being neglected ?

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

15. Proposed by M. A. GRUBER, M. A., War Department, Washington, D. C.

(*a*) The *difference* of two *odd* squares is always divisible by 8. Corollary: Every odd square is of the form $8a+1$

(*b*) The *sum* of two *odd* squares is two times an *odd* number.

I. Solution by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

(*a*) Every odd number is either of the form $4m+1$ or of the form $4m+3$.

$$(4m+1)^2=16m^2+8m+1=8(2m^2+m)+1;$$

$$(4m+3)^2=16m^2+24m+1=8(2m^2+3m)+1.$$

Hence every odd square is of the form $8a+1$, and any two odd squares may be represented by $8p+1$ and $8q+1$; their difference is $8p-8q=8(p-q)$.

(*b*) $(8p+1)+(8q+1)=8p+8q+2=2(4p+4q+1)$

$$=2[2(2p+2q)+1]=2[4(p+q)+1].$$

PROBLEMS.

22. **Proposed by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.**

Decompose into the sum of two squares the product $5\times13\times61$.

23. **Proposed by J. M. COLAW, A. M., Principal of High School, Monterey, Virginia.**

Find three positive integral numbers such that the *product* of the first and the sum of the others is a square and the *sum* of their cubes is a square.

24. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

Solve *generally:* The sum of the cubes of n consecutive numbers is a square. Determine the numbers, when $n=2$, $n=3$, $n=4$, and $n=5$.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

12. **Proposed by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.**

A large plane area is ruled by two sets of parallel eqidistant straight lines, the one set perpendicular to the other. The distance between any two lines of the first set is a; the distance between any two lines of the second set is b. If a regular polygon of $2n$ sides be thrown at random upon this area, find the chance that it will fall across a line, the diameter of the circum-circle of the polygon being less than a or b.

I. Solution by the PROPOSER.

Let $2r$ be the length of the diagonal of the polygon, θ its inclination to the length a.

Then if the centre of the polygon falls anywhere on the area $\{ab-(a-2r\cos\theta)(b-2r\cos\theta)\}$ it will fall across a line.

The limits of θ are 0 and $\frac{\pi}{2n}$.

$$\therefore \text{ Chance}=p=\frac{\int_0^{\frac{\pi}{2n}}\{ab-(a-2r\cos\theta)(b-2r\cos\theta)\}d\theta}{\int_0^{\frac{\pi}{2n}}ab\,d\theta}$$

$$p=\frac{2r(a+b)\sin\frac{\pi}{2n}-\frac{\pi r^2}{n}-r^2\sin\frac{\pi}{n}}{\frac{\pi ab}{2n}}$$

$$= \frac{4r(a+b)n \sin\frac{\pi}{2n} - r^2(2\pi + 2n \sin\frac{\pi}{n})}{\pi ab}.$$

If $n=2$, $p=\dfrac{4r(a+b)\sqrt{2}-r^2(2\pi+4)}{\pi ab}$.

Let $l=4rn \sin\frac{\pi}{2n}$ be the perimeter of the polygon.

Then $p=\dfrac{(a+b)l-2\pi r^2-rl \cos\frac{\pi}{2n}}{\pi ab}$.

Let b be infinite.

Then $p=\dfrac{l}{\pi a}=\dfrac{l}{l'}$, where l' is the perimeter of the circle having a for its diameter.

Excellent solutions were received from *Professors Matz and Draughon*. Their solutions may appear in January Number.

PROBLEMS.

22. Proposed by ALTON L. SMITH, Instructor in Drawing, Polytechnic Institute, Worcester, Mass.

In a series of counts of the votes on a legislative act relative to the city of Worcester, the followidg results were obtained:

	YES.	NO.
1st count	5566	5511
2nd "	5519	5558
3d "	5546	5517
4th "	5512	5551
5th "	5512	5541

What is the probability that the last count (5th) is correct?

23. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the average area of all the triangles that can be drawn *perpendicular-sided* to a given plane scalene triangle.

EDITORIALS.

AN exhaustive solution to problem 33, Arithmetic Department, was received from A. H. Bell, but too late for credit in the proper place.

D. G. [illegible]ANCE should have been credited with solving problem Arithmetic Department.

WE can furnish a limited number of complete sets of Vol. I, for $2.00.

SEND all subscriptions by Money Order, or Draft, made payable to B. F. FINKEL.

DR. F. P. MATZ says: "Raise the subscription price of the MONTHLY no *sane* person can see nor understand how you can give so much mathematics per year at $2.00. Raise the subscription if you intend to keep the MONTHLY a monthly publication."

WE now ask our contributors to exercise the greatest care in preparing papers, problems, and solutions. Write and punctuate your contributions as you would wish to have them appear in print. This will be a great help to the editors and a source of pleasure to the readers.

PROFESSOR E. P. THOMPSON, of Miami University, Oxford, Ohio, says: "I am getting several mathematical periodicals, some foreign, and yours appears to fill a field that needs occupying....... I think your idea of monthly biographies an excellent one. Also by representing each branch of Elementary Mathematics by sets of problems and answers."

WE have contracted with our publishers to have the MONTHLY mailed to our subscribers between the 20th and 25th of each month, thus avoiding the irregularity that prevailed during the past year. This No. was mailed Jan. 23. The delay was unavoidable. We hope to get Jan. No. out in a few weeks and then have subsequent Nos. to appear within the terms of our contract.

One of the Editors of the Cosmopolitan Magazine is Arthur Sherburne Hardy, Ph D., Professor of Mathematics in Darthmouth College and author of a text-book on Analytical Geometry, on Calculus, and on Quaternions. By special arrangements with the managers of the Cosmopolitan, we are enabled to offer a liberal commission to agents to canvass for that popular Magazine and the MONTHLY. Write to us at Kidder, Mo., for terms.

THERE are numerous typographical errors in the current volume. These are as annoying to the editors as to our readers. But when it is remembered that the JOURNAL is made up and proof read after hearing eight recitations per day it is not surprising that such errors should occur. Besides, unfortunately many parts of pages are printed without being proof-read at all, and the greater part of the JOURNAL is printed after one proof-reading. But errors of this nature we hope to reduce to a minimum, as we have determined to give every page a second proof-reading.

THIS number completes Volume I. of the AMERICAN MATHEMATICAL MONTHLY. It is not for us to praise the merits of our own Journal, though there are a few things we may say without violating any rules of propriety. The JOURNAL was started as an experiment. Had we consulted our financial intrests only not a line would have been printed for we expected to lose some money. This expectation was based upon our own calculations and the experience of other pioneers in Mathematical Journalism. Our expectation in this respect has been fully realized, notwithstanding, that some of our warm friends not only trebled and quadrupled their subscriptions, but also paid for diagrams.

plates, and papers. But while we have realized a financial loss, our subscription list has come up to our estimation. More than this, we have the hearty co-operation of the best Mathematicans in America, which also will insure success to any magazine. These ardent workers are not only contributing articles to the MONTHLY which will be incorporated in the future literature of Mathematics, but are also putting forth a strong effort to extend the circulation of the MONTHLY. By these efforts the MONTHLY is gaining a high place among the magazines of the world. In view of these encouragements we have made arrangements to continue the publication of the MONTHLY, at least, another year, and we invite all of our present subscribers to continue with us and also to secure one or two new subscribers. By proper co-operation the enterprise can be made very successful. There is no reason why every Professor and Teacher of Mathematics in the United States should not become a subscriber. The JOURNAL is open to the publication of articles in every department of mathematics. Subscribers should not complain if each issue is not devoted entirely to the particular matter in which they are interested. Vol. I. contains 444 pages and as this large volume only costs $2 no one should complain because it does not contain more.

A FEW weeks ago we purchased a set (4 Vols.) of the AMERICAN ENCYCLOPÆDIC DICTIONARY. This marvelous work, which was completed in Nov. 1894, contains 4,730 large folio pages. It is bound in substantial cloth and printed on good paper (the same as used in Webster's International) and in clear type. Thousands of copies of this work were sold through the coupon system by a paper in Chicago, the cost of the Dictionary thus being $7.35 unbound, or $10.55 bound in cloth. We shall take pleasure in sending this Dictionary bound in 4 volumes (cloth) to any of our readers on receipt of $6.75; leather binding $8.75. The Dictionary is no reprint, but entirly new and is worth, at least, $40.00. Send in your order at once and secure a dictionary which is only surpassed (if surpassed at all) by the Century Dictionary.

The American Mathematical Monthly in the Fortschritte der Mathematik:

An extract from a letter to Dr. George Bruce Halsted from Dr. Paul Staeckel of the University of Halle may be thus translated:

"HONORED SIR: At the meeting of scientists in Vienna, I met Professor Vasiliev of Kasan, who informed me that you would translate his Address at the Lobachevsky-commemoration. This news was very welcome to me, since I cannot understand Russian.

And now to day I have received this translation of yours, and desire to express my profoundest thanks.

You have, by the translation of this most interesting address established a claim to the thanks of the whole mathematical world!

I am much obliged to you for the AMERICAN MATHEMATICAL MONTHLY, which I follow with interest. I will give an account of it in the next volumne of the *Fortschritte der Mathematik*."

THE MONTHLY will be continued as such the coming year and the price will not be increased. Vol. II. will contain about 400 pages. A number of

interesting papers will appear during the year. Trusting that the aid and sympathy of our friends may be relied upon for the new year and thanking you for past favors, we conclude the Old and begin the New.

BOOKS.

Exercices D'Arithmetique. Enonces et Solutions. Par J. Fitz-Patrick, Ancien Professeur de Mathematiques et Georges Chevrel, Directeur de L' Institution Charlemagne, a Tours. With a preface by M. Jules Tannery, Assistant Instructor at the High Normal School. Large 8vo, paper cover, 484 pages, Price, 10 francs. Paris: A. Hermann.

It is believed that there has never before been a work written of the nature of the one under consideration. The publisher in a letter to us says that he thinks there has never before been a work of this nature published either in Europe or America. The table of contents is as follows: Preface: Chapter I., Preliminary Definitions.—Numeration: Chapter II., Addition and Subtraction: Chapter III., Multiplication: Chapter IV., Division: Chapter V., Divisibility of Numbers: Chapter VI., Common Divisor of Whole Numbers: Chapter VII., Prime Numbers: Chapter VIII., Fractions: Chapter IX., Decimal Fractions and Decimal Numbers: Chapter X., Ratios and Proportions: Chapter XI., Different Systems of Numeration: Chapter XII., Squares and Square Roots: Chapter XIII., Cubes and Cube Roots: Chapter XIV., Progression: Chapter XV., Miscellaneous Questions: Chapter XVI., Elementary Notions on the Theory of Numbers: Note by M. Matrot.

Each problem is completely solved.

To give our readers a better idea of the nature of this interesting work, we will translate a few of the questions.

Question 1.—*To investigate the change a number experiences, when we write one or more zeros between its digits.*

The authors first consider the case where one zero is written between the digits and take as an example, the number 6237. For this they write 62037, and show that $62037=9\times6200+6237$. Next they take 6204 and for it, write 620004. This is equal to $99\times6200+6204$. They next take the case where zeros are written between several digits of a number. As an example, they take 36356 for which 306300506 is substituted. This is shown to be equal to $9\times36350+99\times363000+9\times30000000$.

Question 285.—*A number divisible by 4 is always the difference of the squares of two whole numbers.*

Represent by $4N$ any multiple whatever of 4.

We have $4N=2N+2N=2N+N^2+1-N^2-1+2N$

$$=2N+N^2+1-(N^2-2N+1)$$

$$=(N+1)^2-(N-1)^2.$$

Question 403.—*Demonstrate that if $\varphi(n)$ is a multiple of $(n-1)$, the number n is prime.*

Question 421.—*If a and b are two prime numbers, the expression $a^{b-1}+b^{a-1}-1$ is divisible by ab.*

Question 432. *If the number $2p+1$ is prime, the sum $(1.2.3.4. \ldots p)^2+(-1)^p$ is divisible by $2p+1$.*

These questions are sufficient to indicate the value of this fascinating work. It will be especially interesting to that class of Mathematicians who are fond of studying the curious combinations and properties of numbers. Under the Miscellaneous Questions are discussed, triangular numbers, quadrangular numbers, pentagonal numbers, etc. We take pleasure in recommending this work to all who are fond of this kind of Mathematical investigation. B. F. F.

The Elementary Properties of the Elliptic Functions with Examples. By Alfred Cardew Dixon, M. A., Late Fellow of Trinity College, Cambridge; Professor of Mathematics at Queen's College, Galway. 8vo cloth, 142 pp. Price, $1.25. New York: Macmillan and Co.

The object of this work is to supply the wants of those students who, for reasons connected with examinations or otherwise. wish to have a knowledge of the elements of Elliptic Functions, not including the Theory of Transformations and the Theta Functions. *Preface.*

This little book gives a very full treatment of Elliptic Functions and is worthy a careful reading. B. F. F.

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